COSMETIC SURGERIES AND NON-ORIENTABLE SURFACES

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ABSTRACT. By considering non-orientable surfaces in the surgered manifolds, we show that the 10/3- and $-10/3$-Dehn surgeries on the 2-bridge knot $9_{27} = S(49,19)$ are not cosmetic, i.e., they give mutually non-homeomorphic manifolds. The knot is unknown to have no cosmetic surgeries by previously known results; in particular, by using the Casson invariant and the Heegaard Floer homology.

1. INTRODUCTION

The well-known Knot Complement Conjecture says that: If two knots in the 3-sphere S^3 have homeomorphic complements, then they are equivalent, i.e., there exists a homeomorphism $h : S^3 \to S^3$ which takes one knot to the other. It had been conjectured by Tietze in 1908 [19], and was proved by Gordon and Luecke in their cerebrated paper [5] in 1989. Actually Gordon and Luecke showed that; On a nontrivial knot in S^3, nontrivial *Dehn surgery* never yields S^3; to which the Knot Complement Conjecture is an immediate corollary.

The Knot Complement Conjecture can be generalized as follows.

Oriented Knot Complement Conjecture (Bleiler (Kirby's list Problem 1.81(D) [7])) : If K_1 and K_2 are knots in a closed, oriented 3-manifold M whose complements are homeomorphic via an orientation-preserving homeomorphism, then there exists an orientation-preserving homeomorphism of M taking K_1 to K_2.

This conjecture is equivalent to the following in terms of Dehn surgery.

Cosmetic Surgery Conjecture (Bleiler (Kirby's list Problem 1.81(A) [7])): Two surgeries on inequivalent slopes are never purely *cosmetic*.

Here we say that: two slopes are *equivalent* if there exists a homeomorphism of the exterior $E(K)$ of a knot K taking one slope to the other, and two surgeries on K along slopes r_1 and r_2 are *purely cosmetic* if there is an orientation preserving homeomorphism between $K(r_1)$ and $K(r_2)$, and *chirally cosmetic* if the homeomorphism is orientation reversing.

Toward Cosmetic Surgery Conjecture by topological approach, as a first step, we show the following in this paper:

Theorem. *Let K be the knot in S^3 indicated as 9_{27} in the Rolfsen's knot table, which is a two-bridge knot with the Schubert form $S(49,19)$. Then $K(10/3)$ is not homeomorphic to $K(-10/3)$.*

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We remark that the slopes corresponding to $10/3$ and $-10/3$ are inequivalent. Because, if they were, there must be an orientation reversing homeomorphism on $E(K)$ by [2, Lemma 2], but it is impossible since the knot is not amphicheiral.

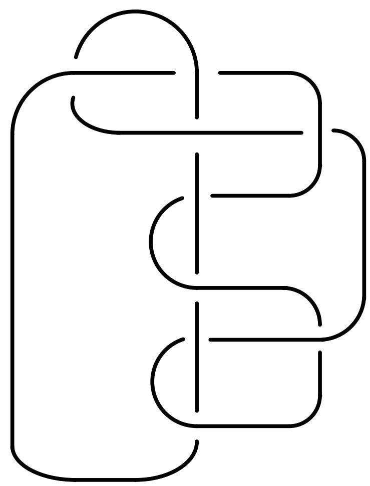

FIGURE 1. The knot 9_{27}

This is very specific calculations for just one example, but previously known results about cosmetic surgery cannot distinguish the pair of manifolds. Also, as far as the author knows, there are no approaches toward Cosmetic Surgery Conjecture by considering non-orientable surfaces, and so, our arguments could give a something new viewpoint.

We here explain our theorem above is in fact contained in the complement of the recent known results, mainly based on *Heegaard Floer technology*, developed and mainly studied by P. Ozsváth and Z. Szabó.

Originally such an approach had started in [14]. After that, together with other invariants of 3-manifolds, Wang showed in [20] that no genus one knot in S^3 admits purely cosmetic surgeries.

Moreover, in [21], Wu proved that for two distinct rational numbers r and r' with $rr' > 0$ and a non-trivial knot K in S^3, $K(r)$ is not orientation preservingly homeomorphic to $K(r')$.

The key ingredient of Wu's proof is using the Casson invariant of 3-manifolds introduced by A. Casson. Actually, Boyer and Lines in [3] had previously proved by using the Casson invariant that a knot K in S^3 satisfying $\Delta''_K(1) \neq 0$ has no cosmetic surgeries. Here $\Delta_K(t)$ denotes the Alexander polynomial for K.

Recently Ni and Wu obtained the following excellent result in [12]; Suppose K is a nontrivial knot in S^3, $r_1, r_2 \in \mathbb{Q} \cup \{0/1\}$ are two distinct slopes such that $K(r_1) \cong K(r_2)$ as oriented manifolds. Then r_1, r_2 satisfy that (a) $r_1 = -r_2$, (b) $q^2 \equiv -1 \mod p$ for $r_1 = p/q$, (c) $\tau(K) = 0$, where τ is the invariant defined by Ozsváth-Szabó defined in [13].

On the other hand, $K = 9_{27}$ is a slice knot, and so $|\tau(K)| \leq g_4(K) = 0$ by [13, Corollary 1.3]. Actually K is a 2-bridge knot, and so, is an alternating knot, for which $\tau(K) = -\sigma(K)/2$ holds, where $\sigma(K)$ denotes the knot signature, as shown in [13, Theorem 1.4]. Obviously we see that $q^2 = 9 \equiv -1 \mod p = 10$.

Moreover the Alexander polynomial for $K = 9_{27}$ is $\Delta_K(t) = -t^3 + 5t^2 - 11t + 15 - 11t^{-1} + 5t^{-2} - t^{-3}$, which implies $\Delta''_K(1) = 0$.

2. Known facts

We here summarize part of known results on cosmetic surgery conjecture. See [2] in detail.

We first remark that the Cosmetic surgery conjecture for "chirally cosmetic" case is not true: there exists counter-example given by Mathieu [9, 10]. Actually, for example, $(18k + 9)/(3k + 1)$- and $(18k + 9)/(3k + 2)$-surgeries on the right-hand trefoil knot $T_{2,3}$ in S^3 yield orientation-reversingly homeomorphic pairs for any non-negative integer k, i.e., the right-hand trefoil admits a chorally cosmetic surgery pairs along inequivalent slopes.

After the discovery of chirally cosmetic surgery on the trefoil by Mathieu, Rong gave in [17] a classification of Seifert knots in closed 3-manifolds (except lens spaces) admitting cosmetic surgeries. Furthermore, Matignon [11] gave a complete classification of non-hyperbolic knots in lens spaces admitting cosmetic surgeries. We remark that the cosmetic surgeries on such knots are all chirally cosmetic.

Concerning hyperbolic knots, Bleiler, Hodgson and Weeks found such an example in [2]]: They showed that there exists a hyperbolic knot which admits a pair of surgeries along inequivalent slopes yielding oppositely oriented lens spaces; $L(49, -19) \leftrightarrow L(49, -18)$ (mirror images). It was then announced in [11] by Matignon (preprint) that there are infinite family extended above.

On the other hand, the following two theorem seems to show that to find cosmetic surgeries are quit hard thing:

Lackenby [8]: Let K be a homotopically trivial knot with irreducible, atoroidal exterior in 3-manifold with $\beta_1 > 0$. Suppose that at least one of the slopes r, r' has a sufficiently high distance with the meridian. Then $K(r)$ and $K(r')$ are orientation preserving homeomorphic if and only if $r = r'$, and are orientation reversing homeomorphic if and only if K is amphicheiral and $r = -r'$.

Bleiler-Hodgson-Weeks [2]: For a hyperbolic knot K, there exists a finite set of slopes E such that if r, r' are distinct slopes outside E, $K(r)$ and $K(r')$ homeomorphic implies that there exists an orientation reversing isometry h such that $h(r) = r'$. In particular, if K is a knot in S^3, then K is amphicheiral and $r = -r'$.

3. Proof

We begin with recalling basic definitions and terminology about Dehn surgery. See [16] in details for example.

A Dehn surgery is the following operation for a given knot K (i.e., an embedded circle) in a 3-manifold M. First to take the exterior $E(K)$ of K (i.e., the complement of an open tubular neighborhood of K in M), and then, glue a solid torus V to $E(K)$. Let γ be the slope (i.e., an isotopy class of non-trivial unoriented simple loop) on the peripheral torus of K in M which is represented by the curve identified with the meridian of the attached solid torus via the surgery. Then, by $K(\gamma)$, we denote the manifold which obtained by the Dehn surgery on K, and call it the 3-manifold obtained by Dehn surgery on K along γ. In particular, the Dehn surgery on K along the meridional slope is called a *trivial* Dehn surgery.

When K is a knot in S^3, by using the standard meridian-longitude system, slopes on the peripheral torus are parametrized by rational numbers with $1/0$. Thus, when a slope γ corresponds to a rational number r, we use $K(r)$ in stead of $K(\gamma)$.

Now let us start to prove our theorem. Let K be the knot in S^3 labeled as 9_{27} in Rolfsen's knot table. We then consider the two surgered manifold $K(-10/3)$ and $K(10/3)$, and show that they are not homeomorphic.

The first key claim is the following.

Claim 1. *If p is even, then the surgered manifold $K(p/q)$ contains a closed non-orientable surface.* □

This immediately follows from the result obtained in [18], and actually claimed and used in [1]. Thus our pair of manifold $K(-10/3)$ and $K(10/3)$ both contain closed non-orientable embedded surfaces.

We now consider the minimal genus of such non-orientable surfaces. Here, by the genus of a non-orientable surface F, denoted by $g(F)$, we mean the number of Möbius bands mutually disjointly embedded in F. Also $\chi(F) = 2 - g(F)$ holds for the Euler characteristic $\chi(F)$ for F.

Among our manifold $K(-10/3)$ and $K(10/3)$, the following holds for $K(-10/3)$.

Claim 2. *The manifold $K(-10/3)$ contains a closed non-orientable surface $\hat{F}_1$ of genus at most 5.*

Proof. As demonstrated in [6], in terms of the continued fractional expansions for the parameter of a given two-bridge knot, we have an algorithm to construct embedded surfaces in the two-bridge knot exterior.

By using that, we see that there exists a non-orientable spanning surface F_1 for K of genus 4 with boundary slope -4, meaning that F_1 has a single boundary component which represents the slope corresponding to -4 on $\partial E(K)$. Also, it can be checked by the Dunfield's program [4], which implements the Hatcher-Thurston's algorithm.

We here note that the distance $\Delta(-4, -10/3)$ between the pair of slopes -4 and $-10/3$ is calculated as $|-4\cdot 3-(-10)\cdot 1| = 2$, where the *distance* of slopes is defined as the minimal intersection number of their representatives, and is calculated by $|ps - qr|$ for the slopes p/q and r/s. See [16] for example.

Since $\Delta(-4, -10/3) = 2$, by adding a Möbius band, equivalently, a converse operation of boundary-compressing, we have a non-orientable surface of genus 5 with boundary-slope $-10/3$. □

On the other hand, the following holds for $K(10/3)$.

Claim 3. *The manifold $K(10/3)$ does not contain closed non-orientable surfaces of genus at most 5.*

Proof. We suppose that $K(10/3)$ contains a closed non-orientable surface $\hat{F}_2$ of genus at most 5, and will find a contradiction. After compressions, if necessary, we can assume that $\hat{F}_2$ is incompressible.

Then, as shown by Przytycki in [15, Proposition 3.3], $\hat{F}_2$ can be isotoped so that $F_2 = \hat{F}_2 \cap E(K)$ is incompressible, boundary-incompressible, and not boundary-parallel properly embedded in $E(K)$, and $\hat{F}_2 \cap V$ is incompressible in the attached solid torus V.

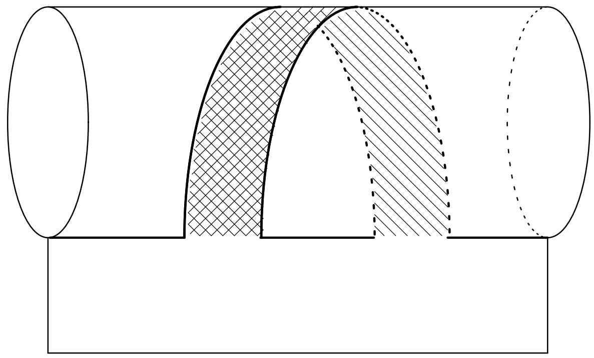

FIGURE 2. Möbius band attaching

For the candidates of $\hat{F}_2$, by using the Dunfield's program, we can verify that there are exactly 8 such surfaces in $E(K)$, and their genera are at least 4. Now, since we are assuming $\hat{F}_2$ of genus at most 5, it implies that the genus $g(F_2)$ of F_2 must be either 4 or 5.

Consider the case where $g(F_2) = 5$. In this case, the Dunfield's program tells us that their boundary-slopes are either -2, 2, 6, or 10. However, since $g(\hat{F}_2) = g(F_2) = 5$, it follows that $\hat{F}_2 - F_2$ must be a disk, which implies the boundary-slope of F_2 must be $10/3$. A contradiction occurs.

Consider the case where $g(F_2) = 4$. In this case, the Dunfield's program tells us that their boundary-slopes are either -8, -4, or 0. Now, since $g(\hat{F}_2) = 5$ and $g(F_2) = 4$, $\hat{F}_2 \cap V$ gives a Möbius band M properly embedded in the attached solid torus V.

Also as shown in [15], this M must be boundary compressible in V. Then, by single boundary-compression on M in V, we have a non-orientable incompressible, boundary-compressible surface F_2' properly embedded in $E(K)$ with boundary-slope $10/3$. This implies that $\Delta(r_2, 10/3) = 2$ must hold for the boundary slope r_2 of F_2.

However, since r_2 must be either -8, -4, or 0, it follows that $\Delta(r_2, 10/3) \neq 2$. Again a contradiction occurs. □

These claims show that the pair of manifolds $K(-10/3)$ and $K(10/3)$ are not homeomorphic.

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